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Waves in meandering streams

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Free-surface and internal stationary waves in a meandering stream are treated, and analytical solutions given. It is shown that for each category there is an infinite number of Froude numbers, depending on the wavenumber of the meander, at which resonance occurs, and the amplitude of one of the wave components becomes infinite, according to the linear theory. These critical Froude numbers are interpreted physically. Furthermore, variable depth is treated for the case of free-surface waves, and in this treatment it is shown, incidentally, how the eigenvalues of a singular differential equation can be found under the requirement that the eigenfunction be non-singular.

Finally, an attempt is made to explain the self-induced, non-stationary waves in water flowing between corrugated vertical walls, found by Binnie (1960), by an instability mechanism proposed by Yih (1976). There is strong evidence that this mechanism is at work, at least when a sloshing mode is involved in the wave-triad interaction.

1. Introduction

When there is need to transport water in open channels from one location to another in a mountainous region, these channels often wind their way more or less along the contour lines of the terrain, and it has been observed that at a certain speed of flow waves of large amplitude form, endangering the unpaved part of the sidewalls. In another practice of hydraulic engineering, water is allowed to shoot at high speed down spillways on the side of a dam. At the entrance of the spillway there is necessarily a contraction, creating violent waves of a diamond pattern, which are obviously undesirable. Hence waves in meandering streams or in channels of variable width are of practical importance. Yet, while the general topic of waves has of late received much attention from research workers, especially that part of it which has to do so with solitons and the inverse-scattering theory, waves in meandering or bulging and contracting streams have seldom been treated. One possible reason for this is perhaps that the problem is not very tractable at first sight.

One of the few papers on the aforementioned problem extant in the literature is that of Binnie (1960), who observed self-induced waves in a conduit with corrugated walls, with longitudinal wavelengths which are an integral multiple of the wavelength of the wall corrugation, and with transverse wavenumbers as well. Binnie gave a brief analysis of slanted waves in otherwise *quiet* water in a rectangular channel with straight sidewalls, which serves to organize his experimental results. His analysis confirmed the existence of the waves propagating upstream, which he observed. But it did not explain how these waves arose. It will be shown at the end of this paper that these progressive waves are induced by an instability mechanism proposed by Yih (1976), originally for the instability of gravity waves in water flowing over a wavy bottom, but adaptable to apply to the stationary waves treated here.

In this paper stationary waves in a meandering stream will be analysed and a complete solution given. The waviness of the sidewalls and its consequences will be fully taken into account, and the solution will give the amplitudes of the various components of the slant waves produced by the flow through the wavy channel at any given value of the Froude number, and in particular will give the transverse wavenumber of the dominant wave at that Froude number. As the Froude number approaches any of an infinite number of critical values resonance occurs, and the amplitude of one of the infinite number of wave components approaches infinity. This result will be given a physical interpretation.

In addition, surface waves created by water flowing in a meandering stream of variable depth, as well as internal waves in a meandering channel with vertical sidewalls, are treated in turn, and similar results are obtained. The results on waves in symmetric channels of variable depth can be treated in the same way (Yih 1982).

2. Formulation of the problem

We neglect the effects of viscosity and assume the flow to be irrotational. The velocity components therefore possess a potential ϕ:

$$(u, v, w) = (\phi_x, \phi_y, \phi_z), \tag{1}$$

where the subscripts denote partial differentiation, x, y, and z are Cartesian coordinates, and u, v, and w are measured in the directions of increasing x, y, and z respectively. The fluid being assumed incompressible, the equation of continuity is

$$u_x + v_y + w_z = 0,$$

which, by virtue of (1), gives the Laplace equation

$$\phi_{xx} + \phi_{yy} + \phi_{zz} = 0, \tag{2}$$

which is the differential equation governing the flow. The coordinate z is measured vertically upward from the free surface when there is no flow, and x and y are measured down and across the channel respectively.

Let the displacement of the free surface above its mean position (which is the position it would have if there were no flow) be denoted by ζ, which is a function of x and y only, since the flow under consideration here is steady. Then the kinematic condition for the free surface is

$$u\zeta_x + v\zeta_y = w, \tag{3}$$

and the dynamic condition there is the Bernoulli equation

$$u^2 + v^2 + w^2 + 2g\zeta = \text{constant}. \tag{4}$$

Combining (3) and (4), we have

$$\left(u\frac{\partial}{\partial x} + v\frac{\partial}{\partial y}\right)(u^2 + v^2 + w^2) + 2gw = 0. \tag{5}$$

Let h be the depth of water in the meandering channel when there is no flow, L be the half-width of the channel at some cross-section, and U be the mean velocity at that cross-section. We shall use L as the lengthscale and U as the velocity scale, and define the following dimensionless variables:

$$\left.\begin{aligned} (\hat{x}, \hat{y}, \hat{z}) = \left(\frac{x}{L}, \frac{y}{L}, \frac{z}{L}\right), \quad (\hat{u}, \hat{v}, \hat{w}) = \left(\frac{u}{U}, \frac{v}{U}, \frac{w}{U}\right), \\ \hat{\phi} = \frac{\phi}{UL}, \quad d = \frac{h}{L}, \quad F^2 = \frac{U^2}{gL}. \end{aligned}\right\} \tag{6}$$

Then, after the carets are dropped, (2) retains its form:

$$\phi_{xx}+\phi_{yy}+\phi_{zz} = 0, \tag{7}$$

which we repeat here because it is dimensionless and will be understood to be so when we later refer to (7). Equation (5), after (1) and (6) are used and the carets in (6) are dropped, now has the *dimensionless* form

$$\left(\phi_x\frac{\partial}{\partial x}+\phi_y\frac{\partial}{\partial y}\right)(\phi_x^2+\phi_y^2+\phi_z^2)+2F^{-2}\phi_z = 0. \tag{8}$$

The boundary condition at the bottom of the channel is

$$\phi_z = 0 \quad (z = -d), \tag{9}$$

and the condition at the vertical walls is

$$\phi_n = 0, \tag{10}$$

where n is measured in a direction normal to the vertical walls bounding the stream. Equations (7)–(10) constitute the differential system governing the problem.

3. A transformation for the meandering

So far we have not taken into account the meander of the stream. This will be represented by the conformal mapping

$$x+\mathrm{i}y = \alpha+\mathrm{i}\beta+\mathrm{i}a\cos k(\alpha+\mathrm{i}\beta),$$

or

$$\left.\begin{aligned} x &= \alpha+a\sin k\alpha\sinh k\beta, \\ y &= \beta+a\cos k\alpha\cosh k\beta, \end{aligned}\right\} \tag{11}$$

where a is an amplitude, and k is a wavenumber of the meander. From (11) we obtain the Jacobian

$$J \equiv \frac{\partial(x,y)}{\partial(\alpha,\beta)} = 1+2ak\cos k\alpha\sinh k\beta+a^2k^2(\sinh^2 k\beta+\sin^2 k\alpha). \tag{12}$$

The transformation (11) gives the meander of the stream: the boundaries of the stream given by $\beta = \pm 1$ are sinuous. Of course other representations are possible, but (11), being conformal, makes subsequent calculations much simpler, and, among the conformal mappings that can possibly represent the meander, it is the simplest.

In terms of α and β (instead of x and y), (7) and (8) become

$$\frac{1}{J}(\phi_{\alpha\alpha}+\phi_{\beta\beta})+\phi_{zz} = 0, \tag{13}$$

$$\frac{1}{J}\left(\phi_\alpha\frac{\partial}{\partial\alpha}+\phi_\beta\frac{\partial}{\partial\beta}\right)\left[\frac{1}{J}(\phi_\alpha^2+\phi_\beta^2)+\phi_z^2\right]+2F^{-2}\phi_z = 0. \tag{14}$$

Equation (9) remains unchanged, but (10) now has the form

$$\phi_\beta = 0 \quad (\beta = \pm 1). \tag{15}$$

The governing system now consists of (13), (14), (9) and (15).

4. Solution of the problem

Recalling that a in (11) is the amplitude of the meander and therefore the amplitude of the waves produced by it, we expand ϕ in a power series in a:

$$\phi = \phi_0 + a\phi_1 + a^2\phi_2 + \ldots. \tag{16}$$

It is evident that

$$\phi_0 = \alpha, \tag{17}$$

which says in effect that in the absence of meander the flow is just unidirectional and uniform flow in a straight channel. Substituting (16) and (17) into (13) and (14), and carefully sorting out the terms of first order in a, we obtain

$$\phi_{1\alpha\alpha} + \phi_{1\beta\beta} + \phi_{1zz} = 0, \tag{18}$$

$$\phi_{1\alpha\alpha} + F^{-2}\phi_{1z} = -k^2 \sin k\alpha \sinh k\beta. \tag{19}$$

The solution for ϕ_1 satisfying (9) and (15) as well as (18) and (19) is, since $\sinh k\beta$ is odd in β,

$$\phi_1 = \sum_{n=1}^{\infty} B_n \sin k\alpha \sin \tfrac{1}{2}(2n-1)\pi\beta \cosh \gamma_n(z+d), \tag{20}$$

where

$$\gamma_n = [k^2 + \tfrac{1}{4}(2n-1)^2\pi^2]^{\frac{1}{2}}, \tag{21}$$

and B_n is given by

$$B_n C_n = -k^2 \int_{-1}^{1} \sinh k\beta \sin \tfrac{1}{2}(2n-1)\pi\beta \, \mathrm{d}\beta = \frac{(-1)^n 2k^3 \cosh k}{\gamma_n^2}, \tag{22}$$

where

$$C_n = -k^2 \cosh \gamma_n d + F^{-2}\gamma_n \sinh \gamma_n d. \tag{23}$$

To order a, then, the *dimensionless* ζ is obtained from (4) (which is in dimensional form) as

$$\zeta = -ak^{-1} \cos k\alpha \sum_{n=1}^{\infty} B_n \gamma_n \sinh \gamma_n d \sin \tfrac{1}{2}(2n-1)\pi\beta. \tag{24}$$

In obtaining (24), we have made use of the result

$$u^2 + v^2 = \frac{1}{J}(\phi_\alpha^2 + \phi_\beta^2)$$

as well as (22), which gives the Fourier coefficients for $\sinh k\beta$. The free-surface displacement given by (24) is shown for one half-wavelength of the meander in figure 1. A perspective view of the free surface is shown in figure 2.

Equations (16), (20), and (23) give the results of the linear theory. Before going on to discuss the next approximation, which takes terms $O(a^2)$ into account, we shall discuss the outstanding features of the results of the linear theory and interpret them in physical terms. First, we see from (22) and (23) that $B_n \to \infty$ when $C_n \to 0$. But for $C_n = 0$

$$F^2 = \frac{\gamma_n^2}{k^2} \frac{\tanh \gamma_n d}{\gamma_n}$$

or

$$\frac{k}{\gamma_n} U = \left(\frac{gh \tanh \gamma_n d}{\gamma_n d}\right)^{\frac{1}{2}}. \tag{25}$$

In (25), k/γ_n is the cosine of the angle between the direction of increasing α and the direction normal to the wave fronts of the slanted waves with wavenumber $\gamma_n d$ (which is the wavenumber non-dimensionalized with the length h instead of the length L), and the right-hand side is the wave speed of waves with the wavenumber $\gamma_n d$. Thus

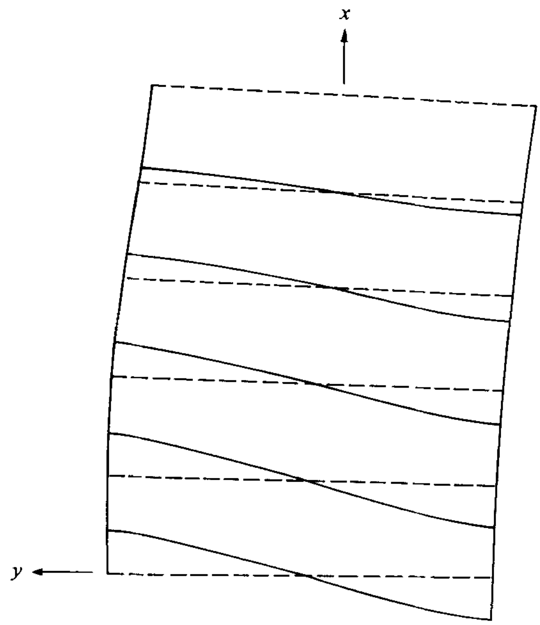

FIGURE 1. Graphs showing the free surface, at $F = 1$, $k = 0.4$, $d = 0.2$. The scale of x is $\frac{1}{4}\pi$ times that of y. The dotted lines are constant-α lines on the undisturbed free surface, for $10k\alpha/\pi = 0$, 1, 2, 3, 4, 5. On the highest one ($4\alpha/\pi = 5$), $\zeta = 0$. The dimensionless ζ/a is plotted above or below the dotted lines. The maximum ζ/a is 0.217, at $\alpha = 0$ and $\beta = 1$. The figure can be extended to $4\alpha/\pi = 10$ by antisymmetry, and then the whole figure can be reflected across the plane $\alpha = 0$ to get the free surface for a whole wavelength.

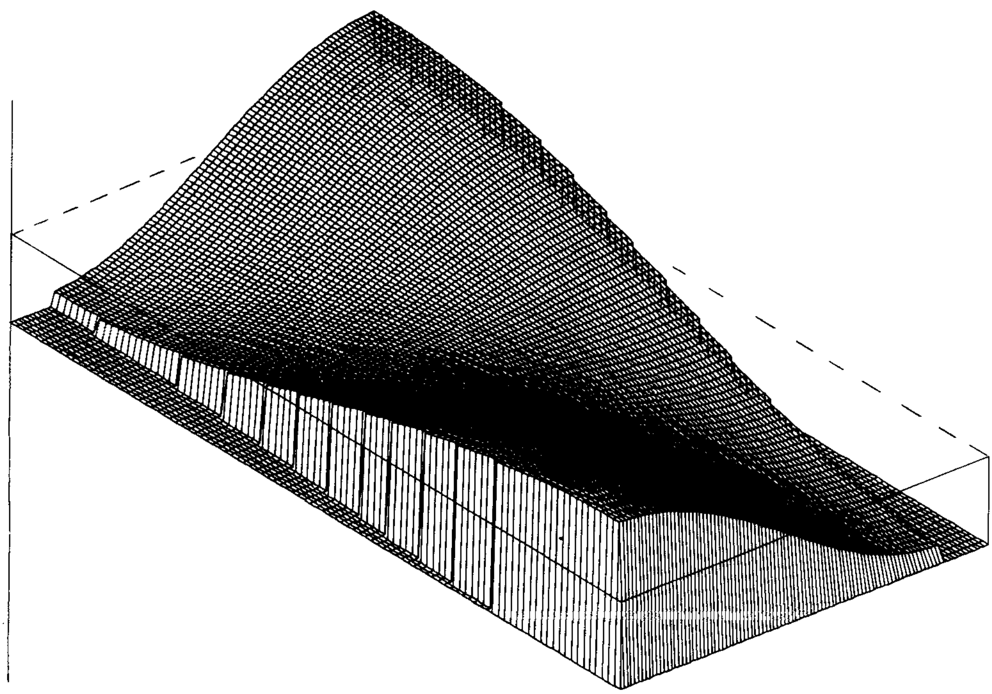

FIGURE 2. Free surface in perspective for half a wavelength, for $F^2 = 0.8$, $k = 0.75$, $d = 0.23$.

the nth critical value of U is such that in the (α, β)-plane its component normal to the fronts of the waves of the wavenumber $\gamma_n d$ is precisely the wave speed of those waves. When U has such a value, there is resonance, and the amplitude of the γ_n waves (with wavenumber γ_n or $\gamma_n d$, depending on the lengthscale used to non-dimensionalize the wavenumber), becomes infinite according to the linear theory. This is reminiscent of the resonance arising from a layer of water flowing over a wavy surface when its speed is equal to the speed of gravity waves with the wavenumber equal to that of the wavy bottom, but now there are an infinite number of critical values for U, and the resonance is recognizable in a somewhat more subtle way.

Higher approximations can be carried out by including the nonlinear terms in (12), (13) and (14), starting with terms $O(a^2)$. We shall not present the rather lengthy calculations, but shall note here that, in the calculation for ϕ_2, new wavenumbers for the α-direction will be obtained. These are zero and $2k$. Calculations to account for terms with these wavenumbers are entirely similar to what has been presented in this section, the only difference being that the equation for ϕ_2, corresponding to (18), will now contain non-homogeneous terms, and the equation for ϕ_2 corresponding to (19) will contain terms with wavenumbers zero and $2k$ for the α-direction. No new critical values for F (or U) will be created by terms of zero wavenumber for the α-direction, but the terms with wavenumber $2k$ (for the α-direction) will give rise to new critical values for F, corresponding to new resonances, which can be interpreted in much the same way as the resonances presented in detail for the basic wavenumber k in this section.

5. Meandering channels of variable depth

Since natural streams have variable depth and are often shallow in the sense that they are much wider than they are deep, we shall treat the case of variable depth with the shallow-water theory, for simplicity. Let α, β, L and F retain the same meanings as before. The h in $d = h/L$ is now variable. Let

$$d = 1-\beta^2. \tag{26}$$

The surface displacement ζ, measured in units of L as before, is again a function of α and β. The total (dimensionless) depth is

$$D = d+\zeta. \tag{27}$$

Then, using dimensionless ϕ, x, and y, the equation of continuity, according to the shallow-water theory, which will be used here, is

$$\frac{\partial}{\partial x}(D\phi_x)+\frac{\partial}{\partial y}(D\phi_y) = 0, \tag{28}$$

and the dimensionless Bernoulli equation written for the free surface is

$$\phi_x^2+\phi_y^2+2F^{-2}\zeta = \text{constant}, \tag{29}$$

where the term ϕ_z^2 is neglected, in consistency with the shallow-water theory. In terms of α and β, (28) and (29) are

$$(d\phi_\beta)_\beta+d\phi_{\alpha\alpha}+\phi_\alpha\zeta_\alpha = 0, \tag{30}$$

$$\frac{1}{J}(\phi_\alpha^2+\phi_\beta^2)+2F^{-2}\zeta = \text{constant}. \tag{31}$$

Using (12), (16) and (17), we obtain from (31)

$$\zeta_\alpha = -aF^2\phi_{1\alpha\alpha} - ak^2F^2 \sin k\alpha \sinh k\beta + O(a^2). \tag{32}$$

Substituting (16), (17), (26) and (32) into (30), extracting terms of first order in a, and writing

$$\phi_1 = \sin k\alpha f(\beta), \tag{33}$$

we have

$$[(1-\beta^2)f']' + [k^2F^2 - k^2(1-\beta^2)]f = F^2k^2 \sinh k\beta, \tag{34}$$

where primes indicate differentiation with respect to β. Since k is based on the *horizontal* lengthscale L, it does not have to be small for the shallow-water theory to apply. However, in most applications (to rivers especially) k is fairly small compared with 1.

We shall now consider the *non-singular* solutions (which will serve as the eigenfunctions) of the equation

$$[(1-\beta^2)G']' + [\lambda - k^2(1-\beta^2)]G = 0, \tag{35}$$

which is singular at $\beta^2 = 1$. Let

$$\lambda = \mu_0 + k^2\mu_1 + k^4\mu_2 + \dots,$$
$$G = g_0 + k^2g_1 + k^4g_2 + \dots. \tag{36}$$

Then it is clear that

$$\mu_0 = n(n+1), \quad g_0 = P_n(\beta), \tag{37}$$

where n is an integer and P_n is the nth Legendre polynomial. To find μ_1 and g_1, we obtain from (35) and (36), upon collecting terms of order k^2,

$$\mathrm{L}g_1 \equiv [(1-\beta^2)g_1']' + n(n+1)g_1 = (1-\beta^2)g_0 - \mu_1 g_0. \tag{38}$$

If g_1 is to be non-singular at $\beta^2 = 1$, the right-hand side of (38) must be orthogonal to g_0. This can be seen by multiplying (38) by g_0 and integrating between -1 and 1 with respect to β:

$$\int_{-1}^{1} g_0 \mathrm{L}g_1 \,\mathrm{d}\beta = \int_{-1}^{1} g_1 L g_0 \,\mathrm{d}\beta = 0 = \int_{-1}^{1} (1-\mu_1-\beta^2) g_0^2 d\beta. \tag{39}$$

This determines μ_1, and g_1 is then found from (38). In practice, we use the well-known identities involving Legendre polynomials, such as those on p. 115 of Jahnke & Emde (1945), and find that

$$\mathrm{L}\left\{\frac{1}{6+4n}\left(\beta^2P_n - \frac{2n}{2n-1}\beta P_{n-1}\right)\right\} = -\beta^2P_n + \frac{2n^2+2n-1}{(2n+3)(2n-1)}P_n,$$

so that

$$g_1 = \frac{1}{6+4n}\left(\beta^2P_n - \frac{2n}{2n-1}\beta P_{n-1}\right), \tag{40}$$

$$\mu_1 = 1 - \frac{2n^2+2n-1}{(2n+3)(2n-1)}. \tag{41}$$

We shall now, for clarity, denote the eigenvalue λ and the eigenfunction G for any particular n by λ_n and G_n, which is the nth (non-singular) eigenfunction of (35). Then, realizing that the right-hand side of (34) is odd in β, we write

$$f(\beta) = \sum_{m=1}^{\infty} B_{2m-1} G_{sm-1}. \tag{42}$$

Substituting this into (34), multiplying the result by G_{2p-1}, and integrating, we have, writing n for $2p-1$,

$$B_n(k^2F^2-\lambda_n)\int_{-1}^{1} G_n^2\,\mathrm{d}\beta = F^2k^2\int_{-1}^{1}\sinh k\beta\, G_n\,\mathrm{d}\beta, \tag{43}$$

which determines B_n (for $n = 2p-1$). Equations (43), (36), (37), and (40)–(42) then give the $f(\beta)$ in (33), and

$$\phi = \alpha + a\phi_1$$

gives the solution for a linear theory, which is all one can attempt in the shallow-water theory. In the foregoing we have determined g_n only to $O(k^2)$. It can be determined up to any power of k^2 with a little patience. For $k = \frac{1}{4}$, it is unnecessary to go beyond what has been done here. For $k = \frac{1}{2}$, the error committed in stopping at terms $O(k^2)$ in g_n is at most 6 %.

For $k = \frac{1}{4}$, the integrals in (43), without their coefficients, have been evaluated. If the first integral is denoted by I_1 and the second integral by I_2, then, for $p = 1, 2, 3$ respectively,

$$I_1 = 0.6551,\quad 0.2870,\quad 0.1917;$$

$$I_2 = 0.1662,\quad 0.0005525,\quad 0.003846.$$

For $k = \frac{1}{2}$,

$$I_1 = 0.6208,\quad 0.2955,\quad 0.1993;$$

$$I_2 = 0.3298,\quad 0.0009392,\quad 0.03908;$$

again for $p = 1, 2, 3$ respectively.

The B_n determined by (43) is infinite for k^2F^2 equal to any of the infinitely many eigenvalues λ_n. There is therefore again resonance at these critical values for k^2F^2, and the physical interpretation for these critical values is analogous to that given in the case of vertical sidewalls in §4, but the mathematical arguments supporting this physical interpretation are now not so transparent, though their vestiges are still evident.

6. Internal waves in a meandering stream

If a stream is laden with sediment, or when water with stratified salinity flows during high tide backward from the sea, internal waves will be created if the stream meanders. These waves generally have a larger amplitude than surface waves and, in the case of the sediment-laden stream, may give rise to turbidity spots where the crests of the internal waves come near the free surface (where water is in contact with air).

For simplicity we shall only give the solution for the case of two layers of liquid of equal and constant depth h. The case of unequal constant depths can be treated similarly, with somewhat more complicated results but no additional difficulty whatever.

Let ρ be the density of the lower fluid and ρ' the density of the upper fluid, and let the corresponding velocity potentials be denoted by ϕ and ϕ' respectively. These both satisfy the Laplace equations.

The displacement of the interface of the two fluids will be denoted by ζ. Then (3) holds for the lower fluid, and a similar one holds for the upper fluid:

$$u'\zeta_x + v'\zeta_y = w', \tag{44}$$

where the primes indicate the upper fluid. We shall describe these in dimensional terms until a later time. Denoting the pressure by p, the Bernoulli equations for the two fluids are

$$\tfrac{1}{2}\rho q^2+\rho g\zeta+p = C,$$

$$\tfrac{1}{2}\rho' q'^2+\rho' g\zeta+p = C',$$

$$\tfrac{1}{2}(\rho q^2-\rho' q'^2)+\Delta\rho g\zeta = C-C', \tag{45}$$

where

$$q^2 = u^2+v^2+w^2 = |\text{grad}\,\phi|^2, \tag{46}$$

$$q'^2 = u'^2+v'^2+w'^2 = |\text{grad}\,\phi'|^2, \tag{47}$$

$$\Delta\rho = \rho-\rho'. \tag{48}$$

Combining (3) with (45), we have the interfacial conditions

$$\left(\phi_x\frac{\partial}{\partial x}+\phi_y\frac{\partial}{\partial y}\right)(\rho q^2-\rho' q'^2)+2g\,\Delta\rho\,\phi_z = 0, \tag{49}$$

$$\left(\phi'_x\frac{\partial}{\partial x}+\phi'_y\frac{\partial}{\partial y}\right)(\rho q^2-\rho' q'^2)+2g\,\Delta\rho\,\phi'_z = 0. \tag{50}$$

The other boundary conditions are

$$\phi_n = 0 = \phi'_n \quad \text{at the vertical sidewalls,} \tag{51}$$

$$\phi_z = 0 \quad (z=-h), \quad \phi'_z = 0 \quad (z=h), \tag{52}$$

where n again denotes the distance along the normal to the sidewalls. The last condition in (52) is obtained by treating the free surface as if it were rigid, as one can do if $\Delta\rho$ is small compared with ρ or ρ'.

We now use the dimensionless variables defined by (6) and similar ones for the upper fluid, removing the carets afterwards, and consider a meander described by (11). The differential equation satisfied by ϕ, in dimensionless terms, remains (13), and the corresponding one for ϕ' is

$$\frac{1}{J}(\phi'_{\alpha\alpha}+\phi'_{\beta\beta})+\phi'_{zz} = 0. \tag{53}$$

The dimensionless forms for (49) and (50) are

$$\frac{1}{J}\left(\phi_\alpha\frac{\partial}{\partial\alpha}+\phi_\beta\frac{\partial}{\partial\beta}\right)K+2F_\mathrm{i}^{-2}\phi_z = 0, \tag{54}$$

$$\frac{1}{J}\left(\phi'_\alpha+\phi'_\beta\frac{\partial}{\partial\beta}\right)K+2F_\mathrm{i}^{-2}\phi'_z = 0, \tag{55}$$

where

$$K = \frac{1}{J(\rho+\rho')}[\rho(\phi_\alpha^2+\phi_\beta^2)-\rho'(\phi_\alpha'^2+\phi_\beta'^2)]+\rho\phi_z^2-\rho'\phi_z'^2, \tag{56}$$

and F_i is the interfacial Froude number defined by

$$F_\mathrm{i}^2 = \frac{U^2(\rho+\rho')}{\Delta\rho g L}. \tag{57}$$

Assuming

$$\phi = \alpha+a\phi_1+a^2\phi_2+\ldots, \quad \phi' = \alpha+a\phi'_1+a^2\phi'_2+\ldots,$$

substituting these into (13) and (53)–(55), and sorting out the terms of order a, we have

$$\phi_{1\alpha\alpha}+\phi_{1\beta\beta}+\phi_{1zz} = 0, \tag{58}$$

$$\phi'_{1\alpha\alpha}+\phi_{1\beta\beta}+\phi_{1zz} = 0, \tag{59}$$

$$\frac{1}{\rho+\rho'}(\rho\phi_{1\alpha\alpha}-\rho'\phi'_{1\alpha\alpha})+F_{\mathrm{i}}^{-2}\phi_{1z} = -\frac{\Delta\rho}{\rho+\rho'}k^2\sin k\alpha\sinh k\beta \quad (z=0), \tag{60}$$

$$\frac{1}{\rho+\rho'}(\rho\phi_{1\alpha\alpha}-\rho'\phi'_{1\alpha\alpha})+F_{\mathrm{i}}^{-2}\phi'_{1z} = -\frac{\Delta\rho}{\rho+\rho'}k^2\sin k\alpha\sinh k\beta \quad (z=0). \tag{61}$$

Equations (60) and (61) immediately give

$$\phi_{1z} = \phi'_{1z}, \tag{62}$$

which can be used to replace (61).

The boundary conditions are now

$$\phi_\beta = 0 = \phi'_\beta \quad (b=\pm 1), \tag{63}$$

$$\phi_z = 0 \quad (z=-d), \quad \phi'_z = 0 \quad \left(z = d \equiv \frac{h}{L}\right). \tag{64}$$

These boundary conditions are satisfied by

$$\phi_1 = \sum_{n=1}^{\infty} B_n \sin k\alpha \sin\tfrac{1}{2}(2n-1)\pi\beta\cosh\gamma_n(z+d), \tag{65}$$

$$\phi'_1 = \sum_{n=1}^{\infty} B'_n \sin k\alpha \sin\tfrac{1}{2}(2n-1)\pi\beta\cosh\gamma_n(z-d), \tag{66}$$

where γ_n is still given by (21).

Now it is evident that condition (62) is satisfied by

$$B'_n = -B_n, \tag{67}$$

and it remains only to determine the B_n in (60), which now has the form

$$\phi_{1\alpha\alpha}+F_{\mathrm{i}}^{-2}\phi_{1z} = -\frac{\Delta\rho}{\rho+\rho'}k^2\sin k\alpha\sinh k\beta. \tag{68}$$

Comparing (68) with (19), we see that the present B_n is equal to $\Delta\rho(\rho+\rho')^{-1}$ times the B_n given by (22), if the F in (23) is replaced by F_{i}. The surface displacement ζ is given by (45) in dimensional terms. For the linear theory we can assume

$$C-C' = \tfrac{1}{2}\Delta\rho\, U^2.$$

The critical Fronde numbers are given by

$$F_{\mathrm{i}}^2 = \frac{\gamma_n}{k^2}\tanh\gamma_n d,$$

and this can be interpreted physically as in §4. The discussion for higher approximations is also similar to that for the surface waves treated in §4.

Finally, we note that the theory presented in the foregoing sections is not merely for supercritical flows, and that when k is large waves of large amplitude can occur even at subcritical speeds. The figures given in this paper are for supercritical speeds, for the F would be larger than 1 if it were based on the mean depth, and the pattern agrees qualitatively with that obtained from the classical shallow-water theory at supercritical speed and for vertical sidewalls. But this should not obscure the fact that the present theory is for all Froude numbers, however large or small.

For supercritical flows in curved channels or channel contractions and expansions (all of rectangular cross-sections), the method of characteristics under the assumption of shallow-water theory can be applied. See, for instance, the four excellent papers in an ASCE Symposium (Ippen *et al.* 1951). For corrugated vertical walls of small amplitude, the shallow-water approximation gives a partial differential equation of constant coefficients and of the hyperbolic type, solvable immediately by the method of separation of variables. For subcritical flows the problem is more interesting and the solutions are richer, as this work shows.

7. Explanation for the self-induced waves observed by Binnie

When one makes water flow between two vertical wavy walls, as Binnie (1960) did, stationary waves bound to the wall corrugation are necessarily created, as shown here in §§2–4. These I have called Binnie waves (Yih 1982†). But Binnie also observed self-induced waves propagating upstream. These, and sloshing two-dimensional waves which I think must have also existed in his experiments, are also Binnie waves. It now remains to explain how these unsteady waves, which are either progressive or standing waves, are produced.

Benjamin (1967), in an important paper dealing with the interesting Benjamin–Feir instability of dispersive waves (Benjamin & Feir 1967), said of Binnie's progressive waves: 'I am strongly inclined to believe this is an instance of the type of instability under discussion.' He did not, however, use Binnie's data to support or disprove his claim. It turns out that the mechanism I proposed (Yih 1976) for the instability of gravity waves created by a stream of water flowing over a wavy bottom can be applied to the stationary waves treated in §2–4. This seems quite natural. I shall now study some of Binnie's statements and examine his data in some detail, to show that there is considerable evidence that my instability mechanism may be the cause for Binnie's non-stationary (unsteady) waves.

The wavenumbers γ_n defined by (21) are for a meandering channel. Those for a symmetric channel with wavy walls (or for half of one), which was what Binnie used, are given by (Yih 1982)

$$\gamma_n = (k^2+n^2\pi^2)^{\frac{1}{2}}, \quad n \text{ an integer}. \tag{69}$$

The wavy sides have the basic wavenumber k, and to $O(a)$ the symmetric channel is described by

$$y = \beta + a\cos kh \sinh k\beta, \quad \beta = \pm 1. \tag{70}$$

If we allow a modulation for (70), and replace it by (with a corresponding equation for x)

$$y = \beta + a(\cos kh \sinh k\beta + \epsilon_1 \cos \tfrac{1}{2}k_x \sinh \tfrac{1}{2}k\beta + \epsilon_2 \cos \tfrac{1}{3}k \sinh \tfrac{1}{3}k\beta + \ldots). \tag{71}$$

Then there are stationary waves with wavenumbers

$$(\tfrac{1}{4}k^2+n^2\pi^2)^{\frac{1}{2}}, \quad (\tfrac{1}{9}k^2+n^2\pi^2)^{\frac{1}{2}}, \quad \text{etc.}$$

When $n = 0$, these are

$$\tfrac{1}{2}k, \quad \tfrac{1}{3}k, \quad \tfrac{1}{4}k, \quad \text{etc.}$$

Allowing these, and letting $\boldsymbol{i}$ and $\boldsymbol{j}$ denote the unit vectors in the directions of increasing x and y respectively, we ask the critical question: is there a progressive wavetrain with wavenumbers. (The n below is not the n in (69).)

$$\frac{k}{m}\boldsymbol{i} \pm n\pi\boldsymbol{j}, \quad m, n \text{ integers}, \tag{72}$$

† The B_0 given in that paper is twice as large is it should be.

and a frequency σ (not Binnie's notation), and a transverse oscillation (which is a standing wave) with the wavenumbers

$$\pm n\pi \boldsymbol{j} \tag{73}$$

and the *same* frequency σ? If so, taking either the + or − sign in (72) and (73), and labelling the wavenumbers $\boldsymbol{k}_1$ and $\boldsymbol{k}_2$ and their frequencies σ_2 and σ_1 $(=\sigma_2)$ respectively, we have the satisfaction of the resonance conditions specified by Yih (1976), whose analysis can be extended to the two-dimensional wavenumber space under discussion here. These conditions are

$$\boldsymbol{k}_1 - \boldsymbol{k}_2 = \frac{k}{m}\boldsymbol{i}, \tag{74}$$

$$\sigma_1 - \sigma_2 = 0. \tag{75}$$

The conditions are the same as found by Phillips (1960, 1961) for interaction of wave triads, but, in the case when one wavetrain is created by fluid flow over a wavy boundary, provide the conditions of instability of that wavetrain.

Examining Binnie's tables 1 and 2, one sees that wavenumbers (72) exist. Indeed the agreements between the calculated and observed wave velocity relative to the flowing water in these tables confirms the progressive waves as free (i.e. not bound to wall corrugations) waves progressing upstream, with $m = 2$, 3 and 4, and $n = 0$, 1 and 2 in (72), in the cases observed by Binnie. Binnie did not observe standing waves with wavenumbers $\pm n\pi \boldsymbol{j}$, but it did occur to him to calculate the period of the sloshing modes. (Incidentally he assumed the water to be 'deep', which was true for most cases. When the exact formula for the transverse period is used, I find that the figures for the calculated transverse period are somewhat larger in the cases of the smaller water depths, but in general not affecting – and sometimes even improving – the agreement between the transverse period and the observed period in Binnie's tables.) Assuming that such sloshing modes did exist,† there is general agreement between the periods Binnie observed for his progressive waves and the calculated periods for the sloshing modes with waves $\pm n\pi \boldsymbol{j}$. That is to say, (75) is satisfied – nearly if not precisely, and very consistently, for all cases in which $n = 1$ or 2.

For $n = 0$ the instability is more closely of the kind described by Yih (1976) for wavetrains with wavenumbers all in the longitudinal direction, and (74) is replaced by

$$k_1 - k_2 = \frac{k}{m}. \tag{76}$$

k_1 is the wavenumber of waves progressing upstream that Binnie observed, and k_2 is a smaller wavenumber of waves travelling downstream (even relative to the flowing water). I have checked to see whether (75) and (76) are satisfied by the data Binnie furnished for the cases $n = 0$, and found that the satisfaction is not good, though the magnitudes of the quantities checked are not far off. But even Binnie's observed and calculated values for wave velocity relative to water do not show good agreement for $n = 0$ in his table 1. Thus one must consider the cases of zero n not yet satisfactorily studied. For $n = 1$ and 2, I think there is strong but incomplete evidence that the instability mechanism proposed by me was at work in Binnie's experiments.

† Binnie did say '...these measurements (of wavelengths) were difficult, particularly when large, because over an interval of minutes the amplitudes were unsteady, rising and dying away like beats and the lengths but not the periods are variable.' Could the beats be caused by the standing waves of the sloshing modes?

The smaller m is, the more cycles for the instability mechanism to work, and therefore the more manifest the waves observed by Binnie. This seems to be in general true, except that he never observed any case with $m = 1$. I have checked to see how large the water velocity has to be for (75) and (76) to be satisfied (i.e. for cases $n = 0$) when $m = 1$, and have found that the water velocity has to be less than those used by Binnie. I venture to suggest that travelling waves with $n = 0$ and with $m = 1$ in (76) are possible and can be observed under the right circumstances.

Finally, I note that the mechanism of instability mentioned above does exist theoretically, even though it has been only incompletely demonstrated that it was indeed at work in Binnie's experiments.

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